First Order Differential Equations

Separable: $\mathbf{M(x)\,dx = N(y)\,dy}$

Solution: $\displaystyle\int M(x)\,dx = \int N(y)\,dy$

Linear: $\mathbf{y' + p(x)\,y = g(x)}$

Solution: $\displaystyle\mu\,y = \int \mu\,g(x)\,dx$

Integrating Factor: $\mu = e^{\int p(x)\,dx}$

Exact: $\mathbf{M(x,y)\,dx + N(x,y)\,dy = 0}$
where $\frac{\partial}{\partial \mathbf{y}}\mathbf{M\,dy\,dx} = \frac{\partial}{\partial \mathbf{x}}\mathbf{N\,dx\,dy}$

Solution: $f(x,y) = c$ where $\frac{\partial}{\partial x}f = M$, $\frac{\partial}{\partial y}f = N$

$$f = \text{“}\textit{least common sum}\text{”}\begin{cases}\int M(x,y)\,dx\\ \int N(x,y)\,dy\end{cases}$$

(To make a non-exact equation become exact:
$\mu\,M(x,y)\,dx + \mu\,N(x,y)\,dy = 0$
Integrating Factor: $\ln\mu = \int \frac{M_y - N_x}{N}\,dx$
or $\ln\mu = \int \frac{N_x - M_y}{M}\,dy$
(*integrals above must be single variable*))

Homogeneous: $\mathbf{y' = \frac{P(x,y)}{Q(x,y)}}$

P and Q are polynomials in x and y
all $x^n y^m$ have total power $(n+m)$ the same

Multiply: $y' = \frac{P(x,y)}{Q(x,y)} \cdot \frac{\frac{1}{x^{n+m}}}{\frac{1}{x^{n+m}}}$

Substitute: $\left(\frac{y}{x}\right) = v$ and $y' = v + xv'$
(*This converts equation to a separable DE.*)

Bernoulli: $\mathbf{y' + p(x)\,y = q(x)\,y^n}$

Rewrite: $y^{-n}\,y' + p(x)\,y^{1-n} = q(x)$
Substitute: $y^{1-n} = v$ and $y^{-n}\,y' = \frac{1}{1-n}v'$
(*This converts equation to a linear DE.*)

Autonomous: $\mathbf{y' = f(y)}$

$f(y_0) = 0 \Longrightarrow$ equilibrium solution at $y = y_0$
$f(y_0) < 0 \Longrightarrow$ solutions go down at $y = y_0$
$f(y_0) > 0 \Longrightarrow$ solutions go up at $y = y_0$

“unstable equilibrium” = solutions go away
“stable equilibrium” = solutions go towards
“semi-stable equilibrium” = solutions mixed

Second Order Differential Equations

Homogeneous Linear, Constant Coefficients:
$\mathbf{a\,y'' + b\,y' + c\,y = 0}$

Characteristic Eqn: $a\,r^2 + b\,r + c = 0$

Solution depends on the type of roots:

- $r = r_1,\, r_2$ (*real, not repeated*)
  $y = c_1\,e^{r_1 x} + c_2\,e^{r_2 x}$
- $r = \alpha \pm \beta\,i$ (*complex conjugates*)
  $y = c_1\,e^{\alpha x}\cos(\beta x) + c_2\,e^{\alpha x}\sin(\beta x)$
- $r = r_0,\, r_0$ (*repeated root*)
  $y = c_1\,e^{r_0 x} + c_2\,x\,e^{r_0 x}$

Reduction of Order:
$\mathbf{y'' + p(x)\,y' + q(x)\,y = 0}$
with one solution $\mathbf{y_1 = y_1(x)}$ known

Substitute: $y = \cancel{v\,y_1}$
$y' = \cancel{v\,y_1'} + v'\,y_1$
$y'' = \cancel{v\,y_1''} + 2v'\,y_1' + v''\,y_1$
DE becomes: $(2v'\,y_1' + v''\,y_1) + p\,v'\,y_1 = 0$
Separable: $\frac{1}{(v')}\,(v')' = -\left(p + \frac{2\,y_1'}{y_1}\right)$

Undetermined Coefficients:
$\mathbf{y'' + p(x)\,y' + q(x)\,y = g(x)}$
homogeneous solution $\mathbf{y = c_1\,y_1 + c_2\,y_2}$ known

General solution is $y = c_1\,y_1 + c_2\,y_2 + Y_p$
Y_p is a *particular solution*

Find Y_p by guessing a form and then plugging into DE:

- $g = a_0x^n + a_1x^{n-1} + \cdots + a_n$
  $Y_p = x^s(\,A_0x^n + A_1x^{n-1} + \cdots + A_n\,)$
- $g = (a_0x^n + a_1x^{n-1} + \cdots + a_n)\,e^{\alpha x}$
  $Y_p = x^s(\,A_0x^n + A_1x^{n-1} + \cdots + A_n\,)\,e^{\alpha x}$
- $g = (a_0x^n + \cdots + a_n)\,e^{\alpha x}\cos(\beta x)$ or $\sin(\beta x)$
  $Y_p = x^s(\,A_0x^n + \cdots + A_n\,)\,e^{\alpha x}\cos(\beta x)$
  $+x^s(\,B_0x^n + \cdots + B_n\,)\,e^{\alpha x}\sin(\beta x)$

(x^s *is chosen so that* y_1 *and* y_2 *are not terms of* Y_p.)

Variation of Parameters:
$\mathbf{y'' + p(x)\,y' + q(x)\,y = g(x)}$
homogeneous solution $\mathbf{y = c_1\,y_1 + c_2\,y_2}$ known

General solution is:

$$y = y_1\int\frac{-y_2\,g}{W(y_1,y_2)}\,dx + y_2\int\frac{y_1\,g}{W(y_1,y_2)}\,dx$$

Wronskian: $W(y_1,y_2) = y_1\,y_2' - y_2\,y_1'$

Existence and Uniqueness Theorems

First Order, General Initial Value Problem:
$\mathbf{y' = f(x,y), \quad y(x_0) = y_0}$

- Solution exists and is unique if f and $\frac{\partial}{\partial y}f$ are continuous at (x_0,y_0).
- Solutions are defined somewhere inside the region containing (x_0,y_0) where f and $\frac{\partial}{\partial y}f$ are continuous.

Linear Initial Value Problem:
$\mathbf{y' + p(x)\,y = g(x), \quad y(x_0) = y_0}$

- Solution exists and is unique if p and g are continuous at x_0.
- Solution is defined on the entire interval containing x_0 where p and g are continuous.

Note: higher order linear is the same.

Differential Equations as Vibrations

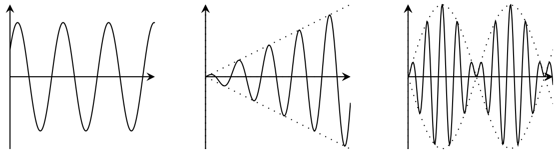
$$\mathbf{m\,y'' + \gamma\,y' + k\,y = F(x)} \qquad \begin{cases} m & \text{mass}\\ \gamma & \text{dampening}\\ k & \text{spring constant}\\ F & \text{forcing function}\end{cases}$$

(electric: $\mathbf{L\,Q'' + R\,Q' + \frac{1}{C}\,Q = E}$)

- (Undamped) natural freq. $\omega_0 = \sqrt{\frac{k}{m}}$
- (Damped) quasi-frequency $\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$

Resonance occurs if forcing freq. ≈ system freq.

Undamped: $\boldsymbol{\gamma = 0}$

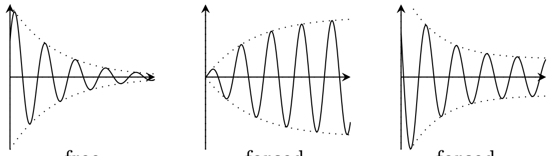


free
$y = R\cos(\omega_0\,x - \delta)$
$R^2 = c_1^2 + c_2^2,\ \delta = \text{Tan}^{-1}\frac{c_2}{c_1}$

forced, resonance

forced, no resonance
beat freq $= \frac{1}{2}\,|\omega_F - \omega_0|$

Damped: $\boldsymbol{\gamma^2 < 4mk}$



free

forced, resonance

forced, no resonance

Not pictured: **overdamped** ($\gamma^2 > 4mk$)
critically damped ($\gamma^2 = 4mk$)

Laplace Transforms

Definition: $\mathscr{L}\{f\} = \int_0^\infty e^{-st}\, f(t)\, ds$

$\mathscr{L}\{y\} = Y,$

$$\mathscr{L}\{y'\} = s\,Y - y(0)$$
$$\mathscr{L}\{y''\} = s^2\,Y - s\,y(0) - y'(0)$$
$$\mathscr{L}\{y'''\} = s^3\,Y - s^2\,y(0) - s\,y'(0) - y''(0)$$

Property: $\mathscr{L}\Big\{\frac{d}{dt} f\Big\} = s\,\mathscr{L}\{f\} - f(0)$

Basic Functions:

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad \mathscr{L}\{\cos(b\,t)\} = \frac{s}{s^2+b^2}$$
$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathscr{L}\{\sin(b\,t)\} = \frac{b}{s^2+b^2}$$
$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathscr{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \quad \text{(Step at } t\!=\!c)$$
$$\mathscr{L}\{\delta_c(t)\} = e^{-cs} \quad \text{(Impulse at } t\!=\!c) \qquad \mathscr{L}\{\delta_c(t)\, f(t)\} = e^{-cs}\, f(c)$$

Exp-Shift and Step-Lag Laws:

$$\mathscr{L}\{e^{at}\, f(t)\} = \ell^a\Big[\mathscr{L}\{f(t)\}\Big] \qquad \begin{cases} \mathscr{L}^{-1}\{F(s)\} = e^{-at}\,\mathscr{L}^{-1}\{F(s-a)\} \\ \mathscr{L}^{-1}\{F(s+a)\} = e^{-at}\,\mathscr{L}^{-1}\{F(s)\} \end{cases}$$

$$\mathscr{L}\{u_c(t)\, f(t)\} = e^{-cs}\,\mathscr{L}\{f(t+c)\} \qquad \mathscr{L}^{-1}\{e^{-cs}\,F(s)\} = u_c(t)\,\ell^c\Big[\mathscr{L}^{-1}\{F(s)\}\Big]$$

ℓ^a is the *lag operator*: $\ell^a\big[F(s)\big] = F(s-a)$ and $\ell^a\big[f(t)\big] = f(t-a)$ (Note: $\ell^a\ell^b = \ell^{(a+b)}$)

Derivative Laws:

$$\mathscr{L}\Big\{\frac{d}{dt}\, f(t)\Big\} = s\,\mathscr{L}\{f(t)\} - f(0) \qquad \mathscr{L}\{t\, f(t)\} = -\frac{d}{ds}\,\mathscr{L}\{f(t)\}$$

Convolutions: $(\mathbf{f} * \mathbf{g})(\mathbf{t})$

Definition: $(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

Property: $\mathscr{L}\{f*g\} = \mathscr{L}\{f\}\,\mathscr{L}\{g\}$

Alternate formula: $f*g = \mathscr{L}^{-1}\Big\{\mathscr{L}\{f\}\,\mathscr{L}\{g\}\Big\}$

Nonhomogeneous Systems of Linear DE

Variation of Parameters: $\vec{\boldsymbol{y}}\,' = \mathbf{A}\,\vec{\boldsymbol{y}} + \vec{\boldsymbol{g}}$

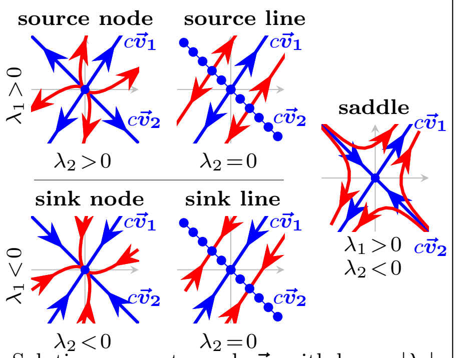
homogeneous solution $\vec{\boldsymbol{y}} = \boldsymbol{\Psi}\,\vec{\boldsymbol{c}}$ known

General solution is:

$$\vec{\boldsymbol{y}} = \boldsymbol{\Psi}\,\vec{\boldsymbol{c}} + \boldsymbol{\Psi}\left(\int \boldsymbol{\Psi}^{-1}\,\vec{\boldsymbol{g}}\, dt\right)$$

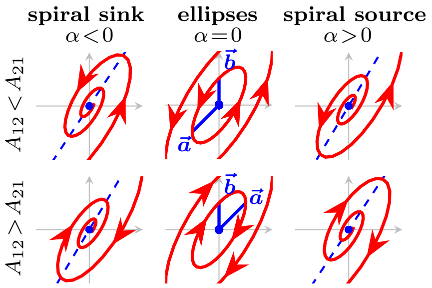
Phase Planes

$\mathbb{R}$ eigenvalues $\lambda_1 \neq \lambda_2$



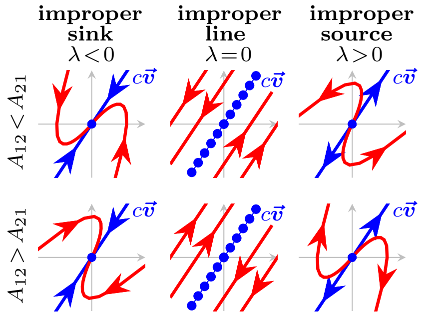
Solutions curve towards $\vec{\boldsymbol{v}}_{\boldsymbol{i}}$ with larger $|\lambda_i|$

$\mathbb{C}$ eig. $\lambda = \alpha \pm \beta\, i$ & $\vec{\boldsymbol{v}} = \vec{\boldsymbol{a}} \pm \vec{\boldsymbol{b}}\, i$



Rotates from $\vec{\boldsymbol{b}}$ to $\vec{\boldsymbol{a}}$ (**CCW** if $A_{12} < A_{21}$)
Stretch; major/minor axes $\vec{\boldsymbol{a}}$ & $\vec{\boldsymbol{b}}$ if $\vec{\boldsymbol{a}} \perp \vec{\boldsymbol{b}}$

Repeated eigenvalues



Rotates **counter-clockwise** if $A_{12} < A_{21}$

Systems of Linear Differential Equations

Constant Coeff. Homogeneous: $\vec{\boldsymbol{y}}\,' = \mathbf{A}\,\vec{\boldsymbol{y}}$

Solution: $\vec{\boldsymbol{y}} = c_1\,\vec{\boldsymbol{y}}_{\mathbf{1}} + c_2\,\vec{\boldsymbol{y}}_{\mathbf{2}} + \cdots = \boldsymbol{\Psi}\,\vec{\boldsymbol{c}}$
where $\vec{\boldsymbol{y}}_{\boldsymbol{i}}$ are **fundamental solutions**
from eigenvalues & eigenvectors as below:

λ-Eigenvector → Fund. Soln. $\vec{\boldsymbol{y}}_{\boldsymbol{i}}$

$$(\mathrm{A} - \lambda I)\,\vec{\boldsymbol{v}} = \vec{\mathbf{0}} \longrightarrow \vec{\boldsymbol{v}}\, e^{\lambda t}$$

Gen. Eigenvect.

$$(\mathrm{A} - \lambda I)\,\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}} \longrightarrow \Big(\vec{\boldsymbol{w}} + \vec{\boldsymbol{v}}\, t\Big)\, e^{\lambda t}$$

Gen.^{2} Eigenvect.

$$(\mathrm{A} - \lambda I)\,\vec{\boldsymbol{u}} = \vec{\boldsymbol{w}} \longrightarrow \left(\vec{\boldsymbol{u}} + \vec{\boldsymbol{w}}\, t + \vec{\boldsymbol{v}}\,\frac{t^2}{2}\right) e^{\lambda t}$$

$\mathbb{C}$ Eigenv. Pair

$$\left.\begin{matrix} \lambda = \alpha \pm \beta\, i \\ \vec{\boldsymbol{v}} = \vec{\boldsymbol{a}} \pm \vec{\boldsymbol{b}}\, i \end{matrix}\right\} \longrightarrow \begin{cases} \Big(\vec{\boldsymbol{b}}\, \cos(\beta t) + \vec{\boldsymbol{a}}\, \sin(\beta t)\Big)\, e^{\alpha t} \\ \Big(\vec{\boldsymbol{a}}\, \cos(\beta t) - \vec{\boldsymbol{b}}\, \sin(\beta t)\Big)\, e^{\alpha t} \end{cases}$$

Note: *solutions above are Imaginary and Real parts of:*

$$(\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\, i)\, e^{(\alpha+\beta\, i)t} = \Big(\vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\, i\Big)\Big(\cos(\beta t) + \sin(\beta t)\, i\Big)\, e^{\alpha t}$$

Fundamental Matrix $\boldsymbol{\Psi}(t) = \begin{bmatrix} | & | \\ \vec{\boldsymbol{y}}_{\mathbf{1}} & \vec{\boldsymbol{y}}_{\mathbf{2}} \\ | & | \end{bmatrix}$

$$\textit{(Real, Non-Defective)} = \begin{bmatrix} | & | \\ \vec{\boldsymbol{v}}_{\mathbf{1}} & \vec{\boldsymbol{v}}_{\mathbf{2}} \\ | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$\textit{(Defective)} = \begin{bmatrix} | & | \\ \vec{\boldsymbol{v}} & \vec{\boldsymbol{w}} \\ | & | \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{\lambda t}$$

$$\textit{(Complex)} = \begin{bmatrix} | & | \\ \vec{\boldsymbol{b}} & \vec{\boldsymbol{a}} \\ | & | \end{bmatrix} \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} e^{\alpha t}$$

Wronskian $\mathrm{W}(t) = \det\Big(\boldsymbol{\Psi}(t)\Big) = \det\left(\begin{bmatrix} | & | & \\ \vec{\boldsymbol{y}}_{\mathbf{1}} & \vec{\boldsymbol{y}}_{\mathbf{2}} & \cdots \\ | & | & \end{bmatrix}\right)$

Exponential $e^{\mathrm{A}t} = \boldsymbol{\Psi}(t)\Big(\boldsymbol{\Psi}(0)\Big)^{-1}$

$$\textit{(Real, Non-Def.)} = \begin{bmatrix} | & | \\ \vec{\boldsymbol{v}}_{\mathbf{1}} & \vec{\boldsymbol{v}}_{\mathbf{2}} \\ | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} | & | \\ \vec{\boldsymbol{v}}_{\mathbf{1}} & \vec{\boldsymbol{v}}_{\mathbf{2}} \\ | & | \end{bmatrix}^{-1}$$
$$= \textit{(etc...)}$$

Init. Value Problem: $\vec{\boldsymbol{y}}\,' = \mathbf{A}\,\vec{\boldsymbol{y}}$ with $\vec{\boldsymbol{y}}(0)$ given.

$$\vec{\boldsymbol{y}} = e^{\mathrm{A}t}\,\vec{\boldsymbol{y}}(0) = \boldsymbol{\Psi}(t)\Big(\boldsymbol{\Psi}(0)\Big)^{-1}\vec{\boldsymbol{y}}(0)$$
$$= \begin{bmatrix} | & | \\ \vec{\boldsymbol{v}}_{\mathbf{1}} & \vec{\boldsymbol{v}}_{\mathbf{2}} \\ | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} | & | \\ \vec{\boldsymbol{v}}_{\mathbf{1}} & \vec{\boldsymbol{v}}_{\mathbf{2}} \\ | & | \end{bmatrix}^{-1} \begin{bmatrix} | \\ \vec{\boldsymbol{y}}(0) \\ | \end{bmatrix}$$
$$= \textit{(etc...)}$$